

1/2 BPS Geometries of M2 Giant Gravitons

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We construct the general 1/2 BPS M2 giant graviton solutions asymptotic to the eleven-dimensional maximally supersymmetric plane wave background, based on the recent work of Lin, Lunin and Maldacena. The solutions have null singularity and we argue that it is unavoidable to have null singularity in the proposed framework, although the solutions are still physically relevant. They involve an arbitrary function $F(x)$ which is shown to have a correspondence to the 1/2 BPS states of the BMN matrix model. A detailed map between the 1/2 BPS states of both sides is worked out.

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1 Introduction

Recently, comprehensive works on 1/2 BPS geometries in the type IIB and 11D supergravities [1], as well as in the six dimensional supergravity have been done [2–4]. (For other related works please see [5].) They are supposed to describe geometries corresponding to 1/2 BPS states of dual CFT in the framework of AdS/CFT correspondence [6, 7]. The story in the case of type IIB turns out to be much interesting, because in this case we know much information on both the geometry side [1, 8] and the corresponding CFT side [9]. The two dimensional phase space of the emergent fermion picture in the 1/2 BPS sector of $\mathcal{N} = 4$ SYM theory is geometrically realized as a surface of specific boundary conditions. Detailed matching between 1/2 BPS operators in $\mathcal{N} = 4$ SYM and the geometries of giant gravitons, in a way suggested by [9, 10], provides another convincing example of AdS/CFT correspondence. It was pointed out that it is even possible to identify the fermion system directly in the supergravity side [12].

Contrary to 1/2 BPS geometries of $AdS_5 \times S^5$ in type IIB, where we only need to solve a linear Laplace equation with a specified boundary condition, the 1/2 BPS geometries in the M-theory are much harder to find because we have to solve a continuum version of nonlinear Toda-like equation [1]. The fact that we again have a boundary plane divided into regions of two different kinds of boundary conditions seems to be similar to the type IIB case, but the nature and even the existence of smooth solutions are not very clear yet. However, for the geometries in the background 11-dimensional plane-wave, in which case there is a translational symmetry along one direction on the boundary plane, it is possible to map the non-linear equation to the linear Laplace equation [1, 13]. In this work, we analyze this linear equation carefully to find solutions of physical relevance.

The solutions we provide in this work depend on an arbitrary function $F(x)$, which we interpret as the number density of 1/2 BPS spherical M2-branes of radius x in the transverse R^3 space of the 11D plane wave background. This identification will be obtained from comparing the charges of the geometries with those of 1/2 BPS solutions in the BMN matrix model. However, we find that the geometries of our interest have null conical singularity. In fact, we will try to give a convincing argument that this type of singularity is inevitable in the proposed ansatz of Lin, Lunin and Maldacena (LLM). We leave it as an open question to discuss the acceptance of this type of singularity. Except this singularity, the resulting geometry seems to have a nice correspondence to the BMN matrix model expectations [7].

After a brief review on the 1/2 BPS geometries in M-theory in the next section, we construct the explicit solutions of 1/2 BPS geometries in the plane-wave background in section 3, followed by explicit examples and analysis in section 4. In section 5, we discuss the singularity and argue that it seems impossible to avoid the singularity in the given ansatz. We compute the charges of the geometries and finally compare them with the BMN matrix model in section 6 and 7.

2 A Review of LLM Construction in M-theory

In this section we review the construction of 1/2 BPS geometries in M-theory following Lin, Lunin and Maldacena (LLM) [1]. M-theory or the eleven dimensional supergravity in the low energy

limit contains two familiar 1/2 BPS objects, M2 and M5 branes. M2 (M5) brane is electrically (magnetically) charged with respect to the three form gauge potential of the eleven dimensional supergravity. The corresponding supergravity solutions interpolate between flat \mathcal{M}_{11} and $AdS_4 \times S^7$ ($AdS_7 \times S^4$) geometries for M2 (M5) branes, which are known to be maximally supersymmetric solutions of M-theory. In addition to this, one additional background is known which is also maximally supersymmetric. This corresponds to the plane wave solution obtained from $AdS_4 \times S^7$ or $AdS_7 \times S^4$ by taking the Penrose limit [14].

Motivated by AdS/CFT correspondence [6] and its extension to plane-wave/CFT correspondence [7], one is interested in the BPS states of the dual CFT obeying $\Delta = J$, where Δ is the conformal dimension of the CFT fields and J corresponds to the $U(1)$ charge in the R symmetry group. Let us consider 1/2 BPS geometries in $AdS_7 \times S^4$. These are associated to the chiral primaries of the (2,0) theory. The chiral primaries of (2,0) theory can be described in terms of two dimensional Young diagrams. These states preserve half of the supersymmetry and are invariant under the symmetry group, $SO(3) \times SO(6) \times R$ of the (2,0) theory on $R_t \times S^5$. In the dual supergravity description, one is interested in the half BPS solutions with this symmetry.

The most general ansatz for the eleven dimensional background consistent with $SO(3) \times SO(6)$ symmetry can be written as [1]

$$\begin{aligned} ds^2 &= e^{2\lambda} \left(\frac{1}{m^2} d\Omega_5^2 + e^{2A} d\Omega_2^2 + ds_4^2 \right) \\ G_{(4)} &= G_{\mu_1 \mu_2 \mu_3 \mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} + F_{\mu_1 \mu_2} dx^{\mu_1} \wedge dx^{\mu_2} \wedge d^2 \Omega_2^2 \end{aligned} \quad (1)$$

where Ω_2^2 and Ω_5^2 being the line elements of the unit two sphere and five sphere respectively and $G_{(4)}$ being the field strength of the three form gauge potential of eleven dimensional supergravity. The Greek indices run over 1, ..., 4 and refer to the four dimensional metric ds_4^2 . λ and A are functions of all the coordinates and m^2 is some fixed parameter independent of the coordinates.

The number of supersymmetries preserved by a background is in one to one correspondence with the number of covariantly constant Killing spinors admitted by the background. To find solutions which preserve some fraction of the supersymmetry one has to solve the Killing spinor equation [15],

$$\nabla_m \xi + \frac{1}{288} (\Gamma_m^{npqr} - 8\delta_m^n \Gamma^{pqr}) G_{(4)npqr} \xi = 0. \quad (2)$$

One can solve the above equation by decomposing the spinor, ξ into spinors of ds_4^2 and seven dimensional spinors and perform the dimensional reduction on S^5 . Further reduction on S^2 gives the equations for the 4-dimensional spinor. One can fix the form of the metric by working out the properties of the Killing spinor bilinears and using Fierz identities. Defining a new coordinate, y in terms of the bilinears, the metric components can be written in terms of a single function, D of spatial coordinates of the four dimensional metric ds_4^2 (for the detailed discussion of the reduction procedure and fixing the form of metric, we refer to the appendix of Ref. [1])

$$\begin{aligned} ds^2 &= -4e^{2\lambda} (1 + y^2 e^{-6\lambda}) (dt + V_i dx^i)^2 + \frac{e^{-4\lambda}}{1 + y^2 e^{-6\lambda}} [dy^2 + e^D (dx_1^2 + dx_2^2)] \\ &\quad + 4e^{2\lambda} d\Omega_5^2 + y^2 e^{-4\lambda} d\Omega_2^2 \end{aligned}$$

$$\begin{aligned}
e^{-6\lambda} &= \frac{\partial_y D}{y(1 - y\partial_y D)} \\
V_i &= \frac{1}{2}\epsilon_{ij}\partial_j D \quad ,
\end{aligned}
\tag{3}$$

where $i, j = 1, 2$. We have not written the expression for the four form $G_{(4)}$, as we shall not need it. It is also determined completely from the knowledge of the function $D(y, x_1, x_2)$. The parameter m in the equation (1) has been set to $m = \frac{1}{2}$. The function $D(y, x_1, x_2)$ satisfies the three dimensional Toda equation

$$(\partial_1^2 + \partial_2^2) D + \partial_y^2 e^D = 0. \tag{4}$$

It can be seen from the metric (3) that the coordinate y is related to the radii of the two sphere, R_2 and of the five sphere, R_5 by the relation $y = R_2 R_5^2/4$. At $y = 0$ either two-sphere or five-sphere shrinks to zero size. The boundary conditions at $y = 0$ are required to be such that the geometry remains non singular when either of the spheres shrinks. In the case where two-sphere shrinks while the radius of five-sphere remains constant, one finds $\partial_y D = 0$ at $y = 0$ and D is independent of y . In the other case where five-sphere shrinks to zero size while the radius of two-sphere remains constant, we have $D \sim \log y$ at $y = 0$. Thus, one can write the boundary conditions at $y = 0$ as

$$\begin{aligned}
\partial_y D &= 0, \quad D = \text{finite}, \quad \text{when } S^2 \text{ shrinks} \\
e^D &\sim y, \quad \text{when } S^5 \text{ shrinks}
\end{aligned}
\tag{5}$$

We shall refer the above boundary conditions as ‘finite type’ and ‘linear type’ respectively as the function e^D becomes finite or goes linearly with y . We shall be interested in solutions of the Toda equation (4) satisfying the above boundary conditions. In general the explicit solutions to Toda equations are not known, but in the particular case when there is an extra spatial isometry (say x_1 in our case) the Toda equation (4) reduces to

$$\partial_2^2 D + \partial_y^2 e^D = 0. \tag{6}$$

By a change of variable*

$$e^D = \rho^2, \quad y = \rho \partial_\rho V, \quad x_2 = -\partial_\eta V \tag{7}$$

the Toda equation (6) can be mapped to the Laplace equation in three dimensions [13]

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho V) + \partial_\eta^2 V = 0. \tag{8}$$

Therefore, the task of solving the Toda equation (6) is reduced to solving the Laplace equation in the new coordinates subject to the boundary conditions (5).

3 M-Theory Giant Gravitons

In this section we are interested in the solutions of the Laplace equation (8) which correspond to giant gravitons in M-theory. (See Ref. [11] for a linearized solution for M-theory giant graviton.)

*Our notation differs from Ref. [1] by $x_2 \rightarrow -x_2$.

In particular we shall be interested in the solutions which are obtained from superposition of the solutions of the Laplace equation. The plane wave background is given by a solution of the Laplace equation [1];

$$V_0 = \rho^2 \eta - \frac{2}{3} \eta^3 \quad . \quad (9)$$

As a warm-up exercise, let us discuss the boundary condition for this solution explicitly. The metric for this solution can be written as

$$ds^2 = -4(\rho^2 + 4\eta^2)dt^2 - 4dx_1 dt + 4(d\rho^2 + d\eta^2) + 4\rho^2 d\Omega_5^2 + 4\eta^2 d\Omega_2^2 \quad . \quad (10)$$

We see that ρ and η are nothing but the radii of the five-sphere and two-sphere respectively. We are interested in the issue of boundary conditions at $y = 0$ in terms of the new coordinates (ρ, η) . They are related to the previous (y, x_2) coordinates by

$$y = 2\rho^2 \eta, \quad x_2 = -(\rho^2 - 2\eta^2) \quad . \quad (11)$$

The boundary at $y = 0$ in the (ρ, η) -plane corresponds to either $\rho \rightarrow 0$ (S^5 shrinking), while keeping η (radius of two sphere) finite or $\eta \rightarrow 0$ (S^2 shrinking), while keeping ρ (radius of five sphere) finite. When the five sphere shrinks one can see that the boundary conditions at $y = 0$ are of the ‘linear type’ that is $e^D \sim y$. In the other case when the two sphere shrinks, the boundary conditions at $y = 0$ are of the ‘finite type’ that is $e^D \sim \text{finite}$.

Now we shall construct solutions which are superpositions of the solutions of Laplace equation (8) upon the plane wave background. Later we shall identify these solutions as half BPS states of BMN matrix model [7]. Solutions to the Laplace equation have several types. There are polynomial type solutions like the case of the above plane wave solution. To recover the plane-wave background asymptotically, they should be absent. Other class of solutions are the superposition of

$$K_0(k\rho) e^{ik\eta}, \quad I_0(k\rho) e^{ik\eta}, \quad (12)$$

or

$$J_0(k\rho) e^{k\eta}, \quad N_0(k\rho) e^{k\eta}, \quad (13)$$

in terms of the Bessel functions. After careful analysis of these types of solutions, it turns out that the relevant one for our application is the solutions involving $K_0(k\rho)$, the zeroth order Bessel function of the imaginary argument, which shows the logarithmic singularity at $\rho = 0$. For this choice, generic solutions to the Laplace equation (8) can be written as

$$V_1 = 2 \int_{-\infty}^{\infty} dk A(k) K_0(k\rho) e^{ik\eta}. \quad (14)$$

Using the integral representation for the Bessel function $K_0(x)$

$$K_0(x) = \int_0^{\infty} dt \frac{\cos tx}{\sqrt{1+t^2}} = \frac{1}{2} \int_{-\infty}^{\infty} dt \frac{e^{itx}}{\sqrt{1+t^2}} \quad , \quad (15)$$

the final form of the solution obtained from superposing on the plane-wave background is

$$V = V_0 + V_1 = \rho^2 \eta - \frac{2}{3} \eta^3 + \int_{-\infty}^{\infty} dt \frac{F(\eta + t\rho)}{\sqrt{1+t^2}} \quad (16)$$

where the function $F(x)$ is the inverse Fourier transform of $A(k)$. In fact, V_1 is the electromagnetic potential of a line charge density $F(\eta)$ distributed along the η -axis.

The other choices of polynomials does not lead to the asymptotically plane wave geometries. One may also show that the choices of I_0 , J_0 and N_0 are not consistent with the asymptotically plane wave geometries with the desired boundary conditions. In section 5, we will actually consider the most general acceptable form of V_1 for the solutions of the Laplace equation, and compare with the properties of the solutions in this section.

Using (7), the expression for y and x_2 may be written as

$$\begin{aligned} y &= 2\rho^2\eta + \rho \int_{-\infty}^{\infty} dt \frac{t}{\sqrt{1+t^2}} F'(\eta + t\rho) \\ x_2 &= -\rho^2 + 2\eta^2 - \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{1+t^2}} F'(\eta + t\rho) \end{aligned} \quad (17)$$

where prime denotes taking derivative with respect to the argument.

In terms of the coordinates (ρ, η) , the expression for metric can be written explicitly as:

$$\begin{aligned} ds^2 &= 4e^{-4\lambda} \left[-W (dt + V_1 dx_1)^2 + \frac{\rho^2}{4W} e^{6\lambda} (dx_1)^2 \right] + \frac{K}{W} e^{2\lambda} (d\rho^2 + d\eta^2) \\ &\quad + 4e^{2\lambda} d\Omega_5^2 + y^2 e^{-4\lambda} d\Omega_2^2 \\ e^{6\lambda} &= W - y^2, \quad V_1 = \frac{1}{K} \frac{\partial y}{\partial \eta}. \end{aligned} \quad (18)$$

where $W = \frac{y\rho K}{2\partial_\rho y}$ and $K = \rho J$, J being the Jacobian of transformation from (y, x_2) coordinates to (ρ, η) coordinates (see below). Unlike the case in terms of the coordinates (y, x_2) , the solution here is no longer implicit. Namely, all the metric components as well as the field strengths can be written explicitly in terms of $F(x)$, though the actual expression may not be much informative.

Let us discuss the boundary conditions for these solutions carefully. To get the right kind of boundary conditions (5) for giant gravitons, we restrict the function $F(x)$ to be an odd function. This choice will be further explained in the later section. In addition, $F(x)$ should be nonnegative for $x \geq 0$. (For the negative $F(x)$, the $y = 0$ line stops somewhere in the η axis. In this case, at the boundary of η axis excluded by the $y = 0$ line, the radii of two and five spheres become finite while ρ goes to zero. For a smooth geometry, one has to utilize x_1 circle but there is no way to match the period of the x_1 circle except for some special cases that are not of our interest.) Other than these restrictions, $F(x)$ can be quite general; $F(x)$ may even be discontinuous as will be illustrated in the next section.

Now let us consider the union I of intervals, $I_n = [a_n, b_n]$ with $0 \leq a_n < b_n$, which is ordered and non overlapping. Namely $b_n < a_{n+1}$. If there are l such intervals, I is given by $I_1 \cup I_2 \cup \dots \cup I_l$. Without loss of generality, one may consider $F(x)$ being nonzero only in $x \in I$. For this setting of $F(x)$, the boundary at $y = 0$ takes in general the following shapes in the (ρ, η) -plane. First it is clear from (17) that $\eta = 0$ solves $y = 0$ as $F(x)$ is chosen to be odd. Furthermore, $\rho = 0$ solves $y = 0$ equation if η does not belong to I . Thus one may see that $\rho = 0$ -line and $\eta = 0$ -line together form some part of the relevant $y = 0$ boundary. The coordinate (ρ, η) is defined in the region

within the first quadrant, i.e. $\rho \geq 0$ and $\eta \geq 0$. Around each interval I_n , there is a larger interval \bar{I}_n ($I_n \subset \bar{I}_n$), in which $y = 0$ boundary is given by $\rho = g_n(\eta) \geq 0$. The intervals \bar{I}_n may overlap with each other generically. If this happens, the overlapped intervals are merged and we may get a new set of interval $\{\bar{I}_n\}$. This new set is constructed such that each interval \bar{I}_n is disjoint with each other and that $\rho = g_n(\eta) \neq 0$ within each interval. It is also possible that the first interval may be extended to the ρ axis, which means $\rho = g_1(0) > 0^\dagger$. In conclusion, the $y = 0$ boundary consists of ρ axis with $\rho > g_1(a_1)$, η axis that does not belong to \bar{I} , and the collection of $\rho = g_n(\eta)$ lines for $\eta \in \bar{I}$. We illustrate the shape of boundary for $l = 3$ case in Fig. 1a by bold line.

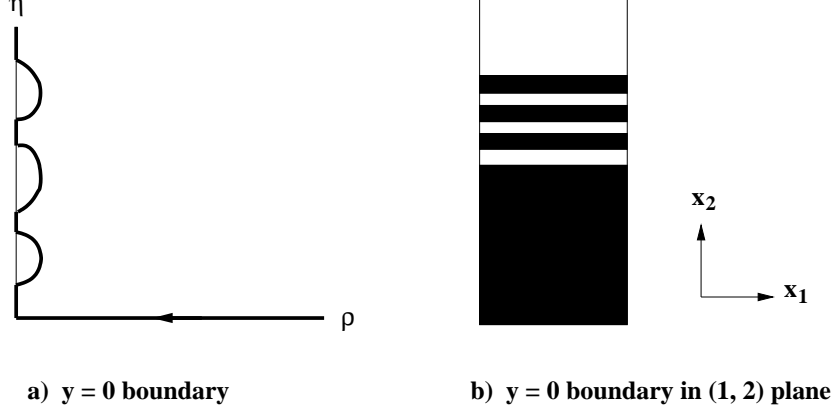


Figure 1: An illustration of the shape of $y = 0$ boundary in (ρ, η) -plane and in (x_1, x_2) for $l = 3$.

Along the boundary in (ρ, η) space, that is, along the $y = 0$ level curve, x_2 has a monotonic behavior. For instance, x_2 in Fig. 1a is monotonically increasing along the boundary in the direction of the arrows. The proof of this follows from the fact that the Jacobian is positive definite for $y \geq 0$, which will be shown shortly, and that y is increasing to the right of the boundary. Then for each interval \bar{I}_n , one has corresponding intervals $\mathcal{I}_n = [\alpha_n, \beta_n]$ in the x_2 coordinate. From the monotonic behavior, the intervals \mathcal{I}_n 's are again ordered and not overlapping with each other. In these intervals as well as the $\mathcal{I}_{\text{plane}} = [-\infty, \beta_0]$ with $\beta_0 = x_2(\eta = 0, \rho = g_1(a_1))$, the finite type boundary condition is realized. $\mathcal{I}_{\text{plane}}$ may be viewed as M2 branes responsible for the plane wave background. The intervals \mathcal{I}_n in the x_2 direction is considered as representing extra M2 giant gravitons. The shape of boundary in the (x_1, x_2) plane is depicted in Fig. 1b for the $l = 3$ case. The number of M2 giant is not given by the area of the strip but given by the area weighted by ρ^2 as will be discussed in the section 6.

Let us give more details for $l = 1$ case where $F(x)$ is non zero only in an interval $[a, b]$. Integrating by parts and Taylor expanding the function $F(x)$, we can write the expression for y (for small ρ) as:

$$y = -2F(\eta) + \frac{\rho^2}{2} \left[\partial_\eta x_2(\rho = 0) + 3\rho^2 g(\eta) + \dots \right] \quad (19)$$

where

$$g(\eta) = \int_{-\infty}^{\infty} dw \frac{F(\eta + w)}{|w|^5}.$$

[†]Note that a_1 is zero in this case.

From this expression we note the following:

i. For $\rho = 0$ and $\eta \in [a, b]$, one can see that $y \neq 0$ and hence $\rho = 0$ and $\eta \in [a, b]$ is excluded from the boundary.

ii. The condition that $y \geq 0$ for infinitesimal ρ further requires $x'_2 \geq 0$ and one can see that the intervals $[\eta_{min}, a]$ and $[b, \eta_{max}]$ are also excluded from the boundary, η_{min} and η_{max} being the roots of $\partial_\eta x_2 = 0$ to the left of a and right of b respectively.

iii. The $\rho > 0$ and $\eta \notin [\eta_{min}, \eta_{max}]$ region is also not a boundary except for $\eta = 0$, which is a trivial boundary with boundary conditions of finite type

$$\rho^2 = e^D \sim \text{finite}$$

iv. For $\rho = 0$ and $\eta \notin [\eta_{min}, \eta_{max}]$ the boundary conditions are of linear type as the function $F(\eta)$ vanishes outside the interval $[\eta_{min}, \eta_{max}]$

$$\rho^2 = e^D \sim y$$

v. For $\rho > 0$ and $\eta \in [\eta_{min}, \eta_{max}]$, the boundary conditions are of finite type as $y = 0$ boundary looks like a bump toward $\rho \geq 0$ region

$$\rho^2 = e^D \sim \text{finite}$$

However, we always find that $\partial_y D = 0$ condition is not satisfied in this part, and the geometry has a null singularity as will be discussed in section 5. Thus a suitable choice of the function $F(\eta)$ (odd and positive in a finite interval) gives the solutions which obey the almost right kind of boundary conditions for a giant graviton except the appearance of null singularity. In the next section we shall discuss this in an explicit example.

By now we have not worried about smoothness of the coordinate transformation from (y, x_2) coordinates to (ρ, η) coordinates. For a well behaving solution in (y, x_2) coordinates one would require that the change of coordinates should be well-defined throughout the region. To analyze this, consider the Jacobian of the transformation $(y, x_2) \rightarrow (\rho, \eta)$

$$J = \frac{\partial y}{\partial \rho} \frac{\partial x_2}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial x_2}{\partial \rho}. \quad (20)$$

Using the definitions (7) of y and x_2 and the equation (8) one finds

$$J = \frac{1}{\rho} \left[\left(\frac{\partial y}{\partial \rho} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right] \quad (21)$$

which is positive semi-definite.

Also we have implicitly assumed that $y \geq 0$. Since y is related to the radii of the two-sphere and five-sphere, one would like to show that radii of two-sphere and five-sphere never become negative. The radii of two-sphere and five-sphere are related to the quantity $e^{6\lambda}$ defined in the metric (18). The requirement for the non-negativity of $e^{6\lambda}$ reads from (18) as

$$K \geq 2 \frac{y}{\rho} \frac{\partial y}{\partial \rho} \quad (22)$$

which is true if the following inequality holds

$$\frac{\partial y}{\partial \rho} \geq \frac{2y}{\rho}. \quad (23)$$

Integrating by parts the expression (17) for y becomes

$$y = 2\rho^2\eta - \rho^2 \int_{-\infty}^{\infty} dx \frac{F(x)}{(\rho^2 + (x - \eta)^2)^{\frac{3}{2}}}. \quad (24)$$

Taking partial derivative with respect to ρ and using the fact that $F(x)$ is an odd function, one gets

$$\frac{\partial y}{\partial \rho} = \frac{2y}{\rho} + 3\rho \int_0^{\infty} dx \left[\frac{\rho^2}{(\rho^2 + (x - \eta)^2)^{\frac{5}{2}}} - \frac{\rho^2}{(\rho^2 + (x + \eta)^2)^{\frac{5}{2}}} \right] F(x). \quad (25)$$

From this, the inequality (23) follows provided $F(x) \geq 0$ for $x \geq 0$. Hence, the non-negativity of y further requires that $F(x) \geq 0$ for $x \geq 0$.

4 Examples of Giant Gravitons

To give a compelling basis to the validity of the solutions we have described, let us analyze an explicit example more carefully. For this purpose, we make the simplest choice of the function $F(x)$, that is, the one with δ -function distribution,

$$F(x) = f_0 (\delta(x - \eta_0) - \delta(x + \eta_0)). \quad (26)$$

It is easy to calculate y and x_2 to find

$$\begin{aligned} y &= \rho^2 \left(2\eta - f_0 \frac{1}{[\rho^2 + (\eta - \eta_0)^2]^{\frac{3}{2}}} + f_0 \frac{1}{[\rho^2 + (\eta + \eta_0)^2]^{\frac{3}{2}}} \right), \\ x_2 &= -\rho^2 + 2\eta^2 + f_0 \frac{\eta - \eta_0}{[\rho^2 + (\eta - \eta_0)^2]^{\frac{3}{2}}} - f_0 \frac{\eta + \eta_0}{[\rho^2 + (\eta + \eta_0)^2]^{\frac{3}{2}}}. \end{aligned} \quad (27)$$

We are interested in the region of $\rho, \eta \geq 0$ in which $y \geq 0$, and the $y = 0$ boundary is of special importance as discussed before. One obvious component of $y = 0$ curve in (ρ, η) plane is η -axis ($\rho = 0$), whereas there may be other components by solving

$$2\eta = f_0 \left(\frac{1}{[\rho^2 + (\eta - \eta_0)^2]^{\frac{3}{2}}} - \frac{1}{[\rho^2 + (\eta + \eta_0)^2]^{\frac{3}{2}}} \right). \quad (28)$$

We will find in the following that there are indeed two additional components from this equation; one is the $\rho \geq 0$ -axis ($\eta = 0$) and the other is an approximate semi-circle (or “bump”) around the point $(\rho, \eta) = (0, \eta_0)$ in the $\rho \geq 0$ region. Let us call the two intersection points of this approximate semi-circle with the η -axis, A and B, whose positions are $(0, \eta_{\min})$ and $(0, \eta_{\max})$. At first sight, this may seem to indicate a problem because $y = 0$ -curve bifurcates at A and B into $\rho = 0$ -curve and the semi-circle. However, we will show that, for $\eta \in [\eta_{\min}, \eta_{\max}]$, we should actually discard $\rho = 0$ segment and take only semi-circle part. This is because the semi-circle turns out to be the correct boundary of $y \geq 0$ region, while $\rho = 0, \eta \in [\eta_{\min}, \eta_{\max}]$ is the boundary of $y \leq 0$ region instead.

Therefore, the whole picture of $y = 0$ boundary for $y \geq 0$ region would be

$$\{\rho \geq 0, \eta = 0\} \cup \{\rho = 0, \eta \in [0, \eta_{\min}] \cup [\eta_{\max}, \infty]\} \cup \{\rho = g(\eta), \eta \in [\eta_{\min}, \eta_{\max}]\} \quad , \quad (29)$$

where in the last component, $\rho = g(\eta)$ is the equation of the (approximate) semi-circle, whose implicit expression is given by (28). In the previous section, we have shown that Jacobian between (ρ, η) and (y, x_2) is positive definite. This implies that x_2 is monotonic along the level contour of $y = 0$ in (ρ, η) plane. Recalling the map $e^D = \rho^2$, we thus have two separate components of $D \sim (\text{finite})$ boundary condition as $y \rightarrow 0$, that is, the first and the last component in (29), while the second component in (29) corresponds to the boundary condition $e^D \sim c y$ as $y \rightarrow 0$. Therefore, this solution is describing one-band excitation of the plane wave background, whose interpretation would be a supergravity realization of 1/2 BPS M2-brane.

We can explicitly confirm the claimed features of $y = 0$ boundary of $y \geq 0$ region in (ρ, η) plane by considering sufficiently large $\eta_0 \gg 1$ with $f_0 \sim \mathcal{O}(1)$. We may reasonably expect these properties (e.g. the existence of the “bump”) not to change for generic η_0 and f_0 , indicating a large degeneracy of solutions for a given topology of one-band excitation. In solving (28), the solution $\eta = 0$ is obvious, and we look for solutions with $\eta \neq 0$. For large η_0 , the right hand side of (28) will be small unless $\eta \sim \eta_0$, and the left hand side tells us that η is small, allowing us to expand the right hand side near $\eta = 0$. This in fact gives us back the $\eta = 0$ solution. However, near $\eta \sim \eta_0$ region, we may keep the first term in the right hand side while the second term is neglected. This gives us the equation

$$\rho^2 + (\eta - \eta_0)^2 = \left(\frac{f_0}{2\eta}\right)^{\frac{2}{3}} \approx \left(\frac{f_0}{2\eta_0}\right)^{\frac{2}{3}} \quad , \quad (30)$$

describing a semi-circle of radius $\left(\frac{f_0}{2\eta_0}\right)^{\frac{1}{3}}$ centered at $(0, \eta_0)$, and the constants $\eta_{\min/\max}$ is given by $\eta_0 \pm \left(\frac{f_0}{2\eta_0}\right)^{\frac{1}{3}}$.

Around this semi-circle, we have

$$y \approx \rho^2 \left(2\eta_0 - f_0 \frac{1}{[\rho^2 + (\eta - \eta_0)^2]^{\frac{3}{2}}} \right) \quad , \quad (31)$$

from which it is clear that the semi-circle bounds the $y \geq 0$ region to its right, while $\rho = 0$, $\eta \in [\eta_{\min}, \eta_{\max}]$ segment is the boundary of a wrong region. It is also seen that for $\eta \in [0, \eta_{\min}] \cup [\eta_{\max}, \infty]$, the $\rho = 0$ line should be taken as the boundary of $y \geq 0$ as in (29). What remains to be shown is that $\rho \geq 0$, $\eta = 0$ is the boundary of the $y \geq 0$ region to its above. Near $\eta = 0$, we have

$$y \approx 2\rho^2\eta \left(1 - \frac{3f_0\eta_0}{[\rho^2 + (\eta_0)^2]^{\frac{5}{2}}} \right) + \mathcal{O}(\rho^2\eta^2) \quad . \quad (32)$$

Therefore, by taking η_0 and f_0 such that $(\eta_0)^4 > 3f_0$, we have $y \geq 0$ above the $\eta = 0$ boundary. Moreover, the arguments in the previous section guarantee that $y \geq 0$ remains true in the whole region bounded by (29) and there is no other boundary. We hope it is by now convincing that (29) is the correct boundary of our solution, and it represents an example of 1/2 BPS one-band excitation of the plane wave background.

It is straightforward to check that these solutions have the correct boundary conditions as we approach $y \rightarrow 0$, that is, $e^D = \rho^2$ behaves either $\sim c y$ or (finite), except the segment $\{\rho = g(\eta), \eta \in$

$[\eta_{\min}, \eta_{\max}]$ on which $\partial_y D \neq 0$. Therefore, we have a band of conical singularity which turns out to be null-like. In seeing this, it is useful to write the resulting metric explicitly after we change the coordinate from (y, x_2) to (ρ, η) as in (18). The reason for this is because the presumably complicated function e^D of (y, x_2) is simply ρ^2 in (ρ, η) plane. Moreover, (ρ, η) is preferred for another reason; they become the usual radial coordinates of R^6 and R^3 in the asymptotic plane-wave background, and this fact will be useful when we calculate charges of our solutions in the later section.

It is easy to generalize the above discussion to find explicit multi-band solutions. By simple superposition of one-band solutions, we have

$$\begin{aligned} y &= 2\eta\rho^2 + \rho^2 \sum_{n=1}^l \left\{ -f_0^n \frac{1}{[\rho^2 + (\eta - \eta_0^n)^2]^{\frac{3}{2}}} + f_0^n \frac{1}{[\rho^2 + (\eta + \eta_0^n)^2]^{\frac{3}{2}}} \right\} , \\ x_2 &= -\rho^2 + 2\eta^2 + \sum_{n=1}^l \left\{ +f_0^n \frac{\eta - \eta_0^n}{[\rho^2 + (\eta - \eta_0^n)^2]^{\frac{3}{2}}} - f_0^n \frac{\eta + \eta_0^n}{[\rho^2 + (\eta + \eta_0^n)^2]^{\frac{3}{2}}} \right\} . \end{aligned} \quad (33)$$

For sufficiently far separated η_0^n and small f_0^n , each n -th term independently generates small semi-circle bump around $\eta \sim \eta_0^n$ in the $y = 0$ boundary of $y \geq 0$ region. The component of $\{\rho \geq 0, \eta = 0\}$ and $\{\rho = 0, \eta \in R^+ - \bigcup_{n=1}^l [\eta_{\min}^n, \eta_{\max}^n]\}$ are as before. Because the behavior near each bump is not affected in an essential way by other bumps, the properties of these solutions, like the nature of singularities, should be same as in the one-band solutions, at least for generic values of η_0^n and f_0^n . As it will become clear in the later section, the charges for multi-band excitations turn out to be additive, conforming to the naive expectation.

We would like to mention that the above solutions obtained from δ function distribution of $F(x)$ are mere examples of vast possible choices of $F(x)$. For instance, we may try a step function: $F(x) = f_0$ for $x \in [a, b]$, $-f_0$ for $x \in [-b, -a]$, and otherwise zero. We have analyzed this case to find one-band excitations for reasonable values of f_0, a and b , and also expect multi-band solutions by superposition of them. We do not believe that different choices of $F(x)$ are related to each other by coordinate reparametrization, and this suggests a huge degeneracy of solutions parameterized by a single function $F(x)$. The physical interpretation of this function will become clear in section 6 when we discuss charges of the solutions.

5 Null Singularity

Our solutions presented in (17) are singular at $y = 0$ if $\eta \in \bar{I}$. In this region of $y = 0$ with $\eta \in I$, the finiteness condition of D is satisfied but $\partial_y D = 0$ is not satisfied. This can be seen as follows. Note that

$$e^{6\lambda} = \frac{y(\rho^2 - y\partial_y \rho^2)}{\partial_y \rho^2} = \frac{y\rho K}{2\partial_\rho y} - y^2 \quad (34)$$

where the first equality is from (3). From the second equality, one may see that $\partial_y D = 2\partial_y \ln \rho = 0$ condition is equivalent to $\partial_\rho y = 0$. However one may see from (25) that $\partial_\rho y$ is finite (and positive) for a finite $\rho > 0$ and for any nontrivial F ($F(x) \geq 0$ when $x \geq 0$). Thus the boundary condition is not satisfied for $\eta \in \bar{I}$ leading to the above mentioned singularities. At the singularities, one finds that $e^{2\lambda}$ as well as the radii of both the two and five spheres are vanishing. These are null-like

conical singularities and not much different from those of extremal branes, whose horizon is singular in general.

Before we discuss the nature of the singularities further, we like to argue that these singularities are in fact unavoidable in our framework for the geometries of 1/2 BPS giant gravitons asymptotic to the plane wave geometry. The most general solution of the Laplace equation that is asymptotic to the plane wave geometry is given by

$$V = \rho^2 \eta - \frac{2}{3} \eta^3 + \int_{-\infty}^{\infty} dx \int_0^{\infty} ds s \int_0^{2\pi} d\theta \frac{F(s, x)}{\sqrt{\rho^2 + s^2 - 2s\rho \cos \theta + (\eta - x)^2}} , \quad (35)$$

where the source $F(\rho, \eta)$ should be distributed outside of $y \geq 0$ region in (ρ, η) -plane. In writing the last part V_1 in (35), we used the fact that V_1 satisfies the three dimensional Laplace equation with the rotational symmetry around η axis, and also the boundary conditions at infinity for the asymptotically plane wave geometries. Let ρ_s be the maximum radius of the nonvanishing source distribution occurring at $\eta = \eta_s$. After a short calculation, we find

$$\frac{\partial y}{\partial \rho} = \frac{2y}{\rho} + \int_{-\infty}^{\infty} dx \int_0^{\infty} ds s \int_0^{2\pi} d\theta \left[\frac{-2s \cos \theta F(s, x)}{\rho Z^{\frac{3}{2}}} + \frac{3(\rho^2 + s^2 - 2s\rho \cos \theta) F(s, x)}{Z^{\frac{5}{2}}} \right] , \quad (36)$$

where $Z = \rho^2 + s^2 - 2s\rho \cos \theta + (\eta - x)^2$. As before, requiring $F(s, x)$ be odd in x and positive definite for $x \geq 0$ for the right shape of the $y = 0$ boundary as in Fig.1a, it can be easily proved that the second term in the above is always positive. This implies that y remains positive in the bulk, and we have conical singularities on the “bump” of $y = 0$ -boundary, exactly same as in the line charge case.

Indeed, the potential (35) is equivalent to the one from a line charge density for outside observers with $\rho > \rho_s$. Using the definition of K_0 , the V_1 part may be rewritten as

$$V_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx \int_0^{\infty} ds s e^{ik(\eta-x)} F(s, x) \int_0^{2\pi} d\theta K_0(kZ(\theta)) \quad (37)$$

where $Z^2(\theta) = s^2 + \rho^2 - 2s\rho \cos \theta$. We then perform the θ integration. Using the formula

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta K_0(kZ(\theta)) = I_0(ks) K_0(k\rho) \quad \text{for } s \leq \rho , \quad (38)$$

one may rewrite (37) as

$$V_1 = \int_{-\infty}^{\infty} dx \frac{\bar{F}(x)}{\sqrt{\rho^2 + (\eta - x)^2}} \quad \text{for } \rho_s \leq \rho , \quad (39)$$

where we define

$$\bar{F}(x) = 2\pi \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy \int_0^{\infty} ds s e^{ik(x-y)} I_0(ks) F(s, y) . \quad (40)$$

Therefore, for the charge distribution localized around the η axis with the cylindrical symmetry, the system may be equivalently described as a line charge density at η axis for the observer outside the maximum of ρ (ρ_s) of the charge distribution. If $F(s, x)$ is odd under x , $\bar{F}(x)$ is also odd. The nice properties derived in the previous paragraph can be seen more easily in this context because we showed in section 2 that they hold for line charge distributions.

We think there is in fact a stronger argument for the singularity on the “bump” of $y = 0$ boundary. Quite generally, suppose we have a solution with the necessary boundary condition on the $y = 0$ curve satisfied, and imagine focusing on the edge point where the boundary condition changes from the linear type to the finite type. Note that for this purpose, it will be sufficient to assume that we can go to the (ρ, η) description only locally around this point, and the conclusion we draw from this is universal in this sense. In (ρ, η) -plane, the $y = 0$ curve leaves the η -axis at this point, and let us consider the expansion of the potential around it. (This kind of point should exist for the giant gravitons unless $y = 0$ line stops at the η axis.) The general argument shows that this kind of point should occur away from the charge distribution.

Taylor expansion around the point $(0, \eta_0)$ should give us polynomial type of solutions of the Laplace equation, which can be constructed order by order by the formula

$$V = \sum_{n \geq 0} F_n(\rho^2) \tilde{\eta}^n \quad , \quad (41)$$

with the recursion relation

$$F_{n+2}(x) = -\frac{4}{(n+2)(n+1)} (x F_n''(x) + F_n'(x)) \quad , \quad (42)$$

where $\tilde{\eta} = \eta - \eta_0$. The sum is actually finite for each independent solution. To describe the $y = 0$ curve leaving η -axis, the lowest order expansion of the potential V looks like

$$V = c_0 + c_1 \tilde{\eta} + c_2 \left(\rho^2 \tilde{\eta} - \frac{2}{3} \tilde{\eta}^3 \right) + c_3 \left(\rho^4 - 8 \rho^2 \tilde{\eta}^2 + \frac{8}{3} \tilde{\eta}^4 \right) + \dots \quad (43)$$

Then $y = \rho \partial_\rho V$ becomes

$$y = 2\rho^2 (c_2 \tilde{\eta} + 2c_3 \rho^2 - 8c_3 \tilde{\eta}^2 + \dots) \quad (44)$$

where c_2 should be nonzero for the boundary condition of linear type. One may see that $y = 0$ curve is described by $\rho = 0$ and the curve of

$$c_2 \tilde{\eta} = -2c_3 \rho^2 + \dots \quad (45)$$

The evaluation of $\partial_\rho y$ leads to

$$\partial_\rho y = 4\rho (c_2 \tilde{\eta} + 4c_3 \rho^2 - 8c_3 \tilde{\eta}^2 \dots) \quad (46)$$

Thus along $y = 0$ with $\rho \neq 0$, we see that $\partial_\rho y \neq 0$ in general. In case $c_3 = 0$, one has to consider the higher order terms but the conclusion is not changing, unless $V(\rho, \eta)$ exactly factorizes into $F(\rho^2)G(\eta)$ where we find no relevant solution.

As mentioned earlier, the only remaining possibility is that $y = 0$ line stops at somewhere in the η axis. This is possible as y works as a coordinate. A possible example arises for the case of $F(x) \leq 0$ for $x \geq 0$. In this case, at the boundary of η axis excluded by the $y = 0$ line, the radii of two and five spheres become finite while ρ goes to zero. To make a smooth geometry, one has to utilize x_1 circle but there is no way to match the period of the x_1 circle except for some special cases that are not of our interest.

This completes our argument for the non-existence of *regular* 1/2 BPS giant solutions that are asymptotic to the plane wave geometries. Therefore in the supergravity description of the ansatz

we are considering, the occurrence of singularities seems inevitable. There are many examples of singular solutions in supergravity theories. As mentioned earlier, the singularities of our solutions are null-like and rather similar to the null singularities occurring in the supergravity description of extremal D-branes. In case of extremal D-branes, we view the D-branes as extra fundamental degrees of freedom corresponding to sources of supergravity fields. Without the source terms describing the D-brane dynamics, the supergravity description would miss some part of the degrees of freedom leading to singularities. Indeed the structure of singularity matches precisely with the Born Infeld type source term for D-branes. Whether M2-brane description [18] of the giant graviton dynamics matches with the singularity structure of the above solution is of particular interest. Further studies are required in this regard.

6 Charges

In this section, we like to identify the charge content of the 1/2 BPS system. This will be essential for the later comparison with the 1/2 BPS states of the matrix model in the later section. First consider the lightcone momentum, p^+ the charge associated with the translation along x_1 direction, which counts the number of D0 branes in the IIA string theory.

Since the solutions are asymptotic to the plane wave, the usual ADM definition of energy and momentum will not work for our case. In Ref. [16], a method of finding the conserved quantities for the background with translational isometry was discussed. Taking x_1 as a relevant isometry direction, one may get the expression for the charge straightforwardly. Here we would like to present an alternative method for computing the charge, which gives precisely the same results. We first go to the IIA string description compactifying the eleven dimensional theory on the circle along x_1 direction. The ten dimensional metric in the string frame is related to the eleven dimensional metric by

$$ds_{11}^2 = e^{-\frac{2\phi}{3}} ds_{10}^2 + e^{\frac{4\phi}{3}} (dx_1^2 + C_\mu dy^\mu)^2 \quad (47)$$

where ϕ is the dilaton field, y^μ is the coordinate for the ten dimensions and C_μ is identified with the R-R one form potential. The integral of $*dC$ over the eight sphere at infinity corresponds to the charge. Working with the following general form of the metric,

$$ds^2 = -h_t dt^2 + 2w dt dx_1 + h_x dx_1^2 + h_{ij} dy^i dy^j, \quad (48)$$

the IIA metric becomes

$$ds_{10}^2 = \sqrt{h_x} \left(-\frac{L}{h_x} dt^2 + h_{ij} dy^i dy^j \right), \quad (49)$$

where we introduce $L = w^2 + h_t h_x$, and $e^{\frac{4\phi}{3}} = h_x$. The only non-vanishing component of C_μ is $C_t = w/h_x$. The expression for the charge reads

$$Q = \frac{1}{2k_{11}^2} \int *dC = \frac{1}{2k_{11}^2} \int_{y=\infty} \sqrt{\det h} [-\partial_i h_x + h_x \partial_i w/w] (1 + h_t h_x/w^2)^{-1/2} h^{ij} n_j r^8 d\Omega \quad (50)$$

where $r^2 = y^i y^i$ ($\sim \rho^2 + \eta^2$ for large r), the integration is over unit eight sphere and we restore $2k_{11}^2$ dependence. The vector n_i that is normal to the eight sphere is defined as $n_i n_i = 1$.

Let us consider the solutions with general $F(x)$, for which the metric components at large r behave as,

$$h_t = O(r^2), \quad w = -2 + O(1/r^5), \quad h_{ij} = \delta_{ij} + O(1/r^5), \quad (51)$$

and

$$h_x = \frac{2y\rho}{e^{4\lambda}\partial_\rho y} \left[\frac{\rho\partial y}{y\partial\rho} - 1 \right] = \frac{3p}{\eta r^5} + \frac{15q}{r^7} + \dots \quad (52)$$

where p and q are defined by

$$p = \frac{1}{2} \int_{-\infty}^{\infty} dx F(x), \quad q = \frac{1}{2} \int_{-\infty}^{\infty} dx x F(x). \quad (53)$$

The first relation in (52) follows directly from the metric (18). The asymptotic expansion of h_x may be obtained from the expansion of y in (24),

$$y = 2\rho^2\eta \left(1 - \frac{2p}{\eta r^3} - \frac{6q}{r^5} + \dots \right). \quad (54)$$

From the asymptotic behavior of the metric components, it is clear that Q diverges once p is non-vanishing. This is one reason why we choose the function $F(x)$ to be odd. We do not claim that the solution is unphysical but only that the charge p^+ is not well-defined for nonzero p . For our choice of odd function $F(x)$, the charge Q becomes

$$Q = \frac{105\omega_8}{2k_{11}^2} q. \quad (55)$$

where ω_8 denotes the volume of the eight sphere of unit size.

Now let us discuss the number of the M2 giants. As discussed in Ref. [1], this may be obtained by evaluating the flux $*G_{(4)}$ through seven cycle (Σ_7) of the five sphere fibered over two-surface which ends on the $y = 0$ region where the five sphere shrinks. This leads to

$$N_2 \sim \frac{1}{\omega_5} \int_{\Sigma_7} *G_{(4)} = \int_{\mathcal{D}} dx_1 dx_2 2e^D|_{y=0}, \quad (56)$$

where \mathcal{D} is the region in the 12 plane where S^2 shrinks. The evaluation of this number for general $F(x)$ appears to be complicated. But when $F(x)$ is small enough and non-vanishing only for large x , N_2 is given simply by

$$\Delta N_2 \sim \int_0^\infty dx F(x), \quad (57)$$

where ΔN_2 counts only the extra M2 branes. When F is small enough, the main contributions comes from the intervals I , for which $2e^D \sim F(\eta)/\eta$ and $dx_2 \sim 4\eta d\eta$. Note that the argument x in $F(x)$ is related to η as one may see for instance in the expression (24). Since η is closely related to the radius of the two-sphere, the natural interpretation of the variable x would be the size of giant M2-brane. In comparison with the BMN matrix model in the next section, this interpretation will be clearer. Since the integral $\int_0^\infty F(x) dx$ counts the number of M2-giants, $F(x)$ may naturally be interpreted as the number density of the giant M2 with respect to its radius x . Since p^+ is proportional to $\int_0^\infty F(x)x dx$, one can see that each M2 should carry the p^+ charge proportional to the size x . Later in the BMN matrix model, one may indeed confirm this prediction.

7 Correspondence to the BMN Matrix Model

The dynamics of the plane wave geometry has a matrix model description. The matrix model may be obtained by the standard method of regularizing the supermembrane action in the maximally supersymmetric plane wave background [7, 17]. The x^- (x_1 in the previous section) is the isometry direction and one is compactifying this direction on a circle of radius R and x^+ (t in the previous section) serves as the lightcone time direction. The conserved momentum of the supermembrane along the x^- direction is quantized $p^+ = p_- = N/R$. In the IIA description, the integer N counts the number of D0 branes. The matrix variables are given by $N \times N$ Hermitian matrices as they represent the strings connecting N D0 branes.

The BPS states of this BMN matrix model have been studied in detail [7, 17, 18]. The 1/2 BPS states corresponding to the giant gravitons are particularly simple. The bosonic part has nine spatial directions described by $N \times N$ hermitian matrices and has $SO(3) \times SO(6)$ R-symmetry.

The 1/2 BPS states are governed by

$$[X_a, X_b] = i\epsilon_{abc} X_c \quad (58)$$

where the indices a, b, c run over 1, 2, 3 and the other directions remain unexcited.

The construction of the general 1/2 BPS states is simple. Let us first take an n dimensional irreducible representation of the $SU(2)$ algebra where n is a positive integer. The Casimir $L^2 = X^2 + Y^2 + Z^2$ is given by $j(j+1) = (n^2 - 1)/4$ where $j = (n - 1)/2$. Thus the radius of the fuzzy sphere corresponding to the giant graviton is given by

$$R_g = \frac{\sqrt{n^2 - 1}}{2}. \quad (59)$$

Then generic 1/2 BPS states are given by any combination of the irreducible representations,

$$T = [n_1] \oplus [n_2] \oplus [n_3] + \cdots \oplus [n_k] \quad (60)$$

where $\sum_{i=1}^k n_i = N$ and we take $n_1 \leq n_2 \leq \cdots \leq n_k$. This may be thought of as representations of two dimensional Young diagrams with total N boxes. Since the gauge equivalent representations are not distinguished, we only care about the combinations and the ordering of the dimensions does not matter. The 1/2 BPS state T may be presented alternatively by

$$T = \sum_{n=1}^{\infty} \oplus f_n [n] \quad (61)$$

where f_n is the multiplicity of the n dimensional irreducible representations. The sum $\sum_{n=1}^{\infty} f_n$ counts the number of total giant gravitons.

The dimension n of the irreducible representation is related to the radius of the fuzzy sphere and it is natural to relate it to the argument x of $F(x)$, the function characterizing our solutions. One may then see the correspondence

$$\int_0^{\infty} dx x F(x) \leftrightarrow \sum_{n=1}^{\infty} n f_n, \quad (62)$$

which counts the number N of the D0 branes. In addition, one has

$$\int_0^\infty dx F(x) \leftrightarrow \sum_{n=1}^\infty f_n, \quad (63)$$

which counts the number of the giant gravitons. Therefore, the physical meaning of $F(x)$ is the density of SU(2)-representations of dimension x in large x continuum limit.

This completes the description of the correspondence. One can see that the data for the giant gravitons in the supergravity description is encoded in the arbitrary function $F(x)$ for the interval $x \geq 0$. The arbitrariness of $F(x)$ corresponds to the arbitrariness of the number of giant graviton for a given dimension n . Hence one may say that the correspondence works for any possible states. Namely any 1/2 BPS states of the matrix theory specified by $\{f_1, f_2, f_3 \dots\}$ has a corresponding supergravity data specified by the function $F(x)$ for $x \geq 0$:

$$\{f_1, f_2, f_3 \dots\} \leftrightarrow F(x) \text{ for } x \geq 0. \quad (64)$$

Thus the counting of states for a fixed charge p^+ is well defined in the matrix model description. For the case of the supergravity, the function $F(x)$ is continuous and the volume of the fluctuation of $F(x)$ for the fixed $p^+ = \int_0^\infty dx x F(x)$ will become zero because we are dealing with infinite dimensional space. The situation here is rather similar to the counting of the fluctuation of the supertube moduli space [19]. The trouble follows from the continuum nature of the description; What we need is a quantization or regularization.

Such a regularization is already achieved in the matrix model description. The state are even protected by the quantum corrections due to the enough number of supersymmetries. The counting of states for a fixed $p^+ = N$ is simple. It is the counting of all possible combination of the states $\{f_1, f_2, f_3, \dots\}$ while fixing the sum $\sum_{n=1}^\infty n f_n$. This problem may be mapped to the problem of the bosonic oscillators with Hamiltonian,

$$H = \sum_{n=1}^\infty n a_n^\dagger a_n \quad (65)$$

where a_n satisfies the commutation relation $[a_n, a_m^\dagger] = \delta_{nm}$ while all the remaining commutators vanish. Then f_n corresponds to the occupation number of the n -th oscillator a_n . The partition function is given by

$$Z(w) = \text{tr} e^{-\beta H} = \prod_{n=1}^\infty (1 - w^n)^{-1} \quad (66)$$

where $w = e^{-\beta}$. It is related to the Dedekind eta function

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^\infty (1 - e^{2\pi i n \tau}) \quad (67)$$

The degeneracy of the state for the fixed N is obtained by

$$d_N = \oint \frac{Z(w)}{w^{N+1}} \frac{dw}{2\pi i}. \quad (68)$$

Evaluating this using the saddle point method, one obtains

$$d_N \sim \frac{1}{4\sqrt{3}N} e^{\pi \sqrt{\frac{2}{3}}N}, \quad (69)$$

for the large N . The entropy is then given by

$$S_N = \ln d_N \sim \pi \sqrt{\frac{2}{3}} N. \quad (70)$$

The counting problem here basically corresponds to finding the possible ways of partitioning N into positive integers.

Since our starting representation in (60) is the representation of the 2d Young diagram, the above chiral boson problem is equivalent to the problem of the 2d Young diagram. Or the above (chiral) boson may be mapped into the NS sector of a complex fermion through the well known bosonization technique. Then the states of NS sector of the fermion is in one to one correspondence with the 2d Young diagram [20]. This 2d Young diagram may be mapped into the $c=1$ matrix model with a harmonic oscillator potential, which is related to the N noninteracting fermions under a harmonic oscillator potential [9]. But from the data of the supergravity, this relation to the fermion seems unmotivated. Namely, we do not find any shape of Fermi liquid in (x_1, x_2) plane of the supergravity description unlike case of the 1/2 BPS geometry of the type IIB string theory [1].

8 Conclusion

The giant gravitons in various $AdS_n \times S^m$ spaces as well as in the plane wave limit have been studied in the context of AdS/CFT correspondence. Recent progress on constructing full back-reacted supergravity geometries of 1/2 BPS giant gravitons, initiated in Ref. [1], gave us much insight on the correspondence including the emergent fermionic system in the type IIB string theory in $AdS_5 \times S^5$. However, given the situation where smooth solutions in the asymptotic plane-wave background in 11D supergravity have not been found, it is an important problem to find solutions in the recent framework of LLM in Ref. [1]. In this work, we performed a careful analysis for this problem, starting from the linear Laplace equation that Ref. [1] proposed. We found a class of solutions that involve an arbitrary function $F(x)$ whose interpretation has a nice correspondence to the BMN matrix model, that is, it is the number density of spherical M2-branes in the transverse R^3 -space with respect to its radius x . However, we found that null-like conical singularities are present along the M2-branes, and tried to argue that this is a necessary consequence of the given ansatz.

There is one possible scenario to understand the nature of troubles that we encounter in this paper. Let us first note that analogous problems arise in the study of charged rotating BPS black holes in four dimensional $N=2$ supergravity theories. All the known solutions have naked singularities. The dual descriptions by strings or D branes are well understood and counting degeneracy is straightforward [21]. Thus the corresponding rotating supergravity black hole solutions should exist. In Ref. [22], it is shown that nonsingular BPS solutions may be constructed by the addition of massive Kaluza Klein fields in such a way that the solutions decompactify near the core to five dimensional black hole solutions with regular horizon. Only asymptotically, they approach four dimensional geometries (times circle). If one tries to eliminate the massive Kaluza Klein modes by smearing over the circle, the geometries become the known four dimensional solutions that are naked singular. If this phenomenon happens in our case too, it implies that the assumption of the

isometry along x_1 circle is too strict. Namely, dropping the ansatz of the isometry and solving the full three dimensional Toda equation, there may be a possibility of resolving the singularities. In this case, our solutions clearly have their physical meaning and validity once away from the singularities; Only the core singularities need to be remedied by uplifting to higher dimensions. Of course our problem is then naturally extended to the problem of finding 1/2 BPS giant gravitons asymptotic to $AdS_4 \times S^7$ or $AdS_7 \times S^4$ geometries. Further studies are required in these directions.

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